Problem Sheet 5

Problem 1

(a) Let $f \in \mathbb{Z}[T]$ be a non-constant polynomial. Show that there are infinitely many primes p such that f has a zero modulo p.
Hint: One possibility is to consider the values $f(n!f(0))/f(0)$ for n large.

(b) Prove that, given an integer N, there are infinitely many primes $p \equiv 1 \mod N$.

Problem 2

Prove that a number field $K/\mathbb{Q}$ is unramified over a prime p if and only if its Galois closure K^{Gal} is.

Problem 3

Let $K/\mathbb{Q}$ be a number field of degree n such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$ is generated by a single element. Let f be the minimal polynomial of α and denote its roots in $\bar{\mathbb{Q}}$ by $\alpha = \alpha_1, \dots, \alpha_n$.

(a) Verify that

$$\frac{1}{f(x)} = \sum_{i=1}^{n} \frac{1}{f'(\alpha_i)(x - \alpha_i)}.$$

Hint: Show that $1 - f(T)\sum_i 1/(f'(\alpha_i)(x-\alpha_i))$ is a polynomial of degree $n-1$ with n roots.

(b) Prove that

$$\mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{\alpha^i}{f'(\alpha)}\right) = \begin{cases} 0 & 0 \le i \le n-2 \\ 1 & i = n-1 \end{cases}.$$

Hint: Expand the identity from a) as power series in $1/x$ and compare coefficients.

(c) Conclude that $\delta_{K/\mathbb{Q}}^{-1}$ is simply $f'(\alpha)^{-1}\mathcal{O}_K$. In particular, it is a principal ideal.

Problem 4

Let A be a Dedekind ring and M a finitely generated torsion-free A-module of generic rank n. (The assumptions mean $am = 0 \Rightarrow a = 0$ for all $0 \neq m \in M$ and $\dim_K(K \otimes_A M) = n$, where $K = \mathrm{Frac}(A)$.)

(a) Prove that M is projective, i.e. that any surjection $A^m \twoheadrightarrow M$ splits.

(b) Show that there is an isomorphism

$$M \cong \mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_n$$

for fractional ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ of A.